THE PRICE OF OBLIVIOUSNESS IN NOISY LINEAR RECONSTRUCTION

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ABSTRACT

An unknown point $x^\star \in \mathbb{R}^d$ is queried through unit linear functionals, each answered with adversarial additive error at most δ. Recent work determined the adaptive minimax error. Its excess above the infinite-query Jung limit $J_d\delta = \sqrt{2d/(d+1)}\,\delta$ vanishes doubly exponentially in the number of queries, and it left the nonadaptive game open. We characterize that game. For every fixed $d \geq 2$, its minimax excess is $\Theta_d(\delta T^{-2/(d-1)})$. Thus adaptivity changes the convergence law from polynomial to doubly exponential. We also resolve the previously open exponential-budget regime in high dimension. If $\log T/d \to \alpha \in (0, \infty)$, then the nonadaptive minimax error satisfies

$$\frac{\mathrm{OPT}_d^{\mathrm{na}}(T, \delta)}{\delta} \longrightarrow \sqrt{\frac{2}{1 - e^{-2\alpha}}}.$$

The upper bounds combine spherical coverings with Jung's theorem. The lower bounds use a common transcript that hides every vertex of a suitably rotated and expanded regular simplex. This construction is what makes classical spherical-cap geometry sharp for point reconstruction. Finally, a batched strategy shows that each additional adaptive batch can double the polynomial exponent, interpolating between the two convergence laws.

1 INTRODUCTION

Consider the following adversarial sensing problem. A reconstructor seeks an unknown $x^\star \in \mathbb{R}^d$. On query $v \in \mathbb{S}^{d-1}$, it receives $r \in \mathbb{R}$ such that

$$|r - \langle v, x^\star \rangle| \leq \delta. \tag{1}$$

After T queries it outputs $\widehat{x}$ and incurs Euclidean error $\|\widehat{x} - x^\star\|_2$. The noise is bounded but otherwise adversarial. This model isolates a basic question in adaptive sensing, robust localization, and reconstruction from privacy-perturbed linear statistics (Haupt et al., 2009; Arias-Castro et al., 2013; Dinur & Nissim, 2003).

Filmus et al. (2026) recently characterized the adaptive game. With $J_d = \sqrt{2d/(d+1)}$, its infinite-query error is $J_d\delta$. For fixed $d \geq 2$, the excess above this limit is $2^{-2^{\Theta_d(T)}}\delta$. The mechanism has two phases. An oblivious spherical net first bounds the feasible diameter. Later queries follow the edges of a near-extremal simplex identified from previous answers. Their paper ends by asking what happens when every direction must be chosen before any answer is observed.

This restriction is not cosmetic. Nonadaptive queries are the natural model when sensing directions must be deployed in parallel, when a fixed query batch is audited before release, or when interaction is unavailable. Classical information-based complexity compares adaptive and nonadaptive recovery at a coarse multiplicative scale. For convex symmetric noisy information graphs, the two errors differ by at most a factor of two (Osipenko, 2019). That comparison cannot see their convergence to the same nonzero limit. The open question is therefore an excess-risk question.

Our results. We determine the nonadaptive excess in fixed dimension and its exact high-dimensional phase curve.

Table 1: Fixed-dimensional convergence. Proper reconstructors output one point. Improper reconstructors predict each future linear functional and have a different limiting benchmark.

Setting	Output	Limiting error	Excess after T queries
Proper, adaptive	point	$J_d\delta$	$2^{-2^{\Theta_d(T)}}\delta$
Proper, nonadaptive	point	$J_d\delta$	$\Theta_d(T^{-2/(d-1)})\delta$
Improper, adaptive	function	δ	$\Theta_d(T^{-2/(d-1)})\delta$

Theorem 1 (Fixed-dimensional nonadaptive rate)**.** *For every fixed $d \geq 2$, there are constants $0 < c_d < C_d < \infty$ and $T_d < \infty$ such that, for all $T \geq T_d$ and $\delta > 0$,*

$$c_d \delta T^{-2/(d-1)} \leq \mathrm{OPT}_d^{\mathrm{na}}(T, \delta) - J_d\delta \leq C_d \delta T^{-2/(d-1)}.$$

Together with the adaptive result of Filmus et al. (2026), Theorem 1 gives a polynomial versus doubly-exponential separation in excess error. Equivalently, reaching excess at most $\epsilon\delta$ requires $\Theta_d(\epsilon^{-(d-1)/2})$ nonadaptive queries, while adaptive query complexity is $\Theta_d(\log\log(1/\epsilon))$.

The improper polynomial rate in the last row is due to Filmus et al. (2026). It approaches the smaller benchmark δ with an output that need not correspond to any point. Its missed-direction lower bound cannot prove a proper error near $J_d\delta$. The simplex transcript in this paper is needed to obtain the middle row.

The dimension dependence contains a second sharp phenomenon. The existing adaptive analysis identifies subexponential and superexponential regimes but does not resolve budgets $T = \exp(\Theta(d))$. For the nonadaptive game we get an exact answer throughout this window.

Theorem 2 (Exponential-budget phase curve)**.** *Let $T = T(d)$ satisfy $\log T/d \to \alpha \in (0, \infty)$. Then for every $\delta > 0$,*

$$\lim_{d\to\infty} \frac{\mathrm{OPT}_d^{\mathrm{na}}(T, \delta)}{\delta} = \sqrt{\frac{2}{1 - e^{-2\alpha}}}.$$

The curve diverges like $1/\sqrt{\alpha}$ as $\alpha \downarrow 0$ and tends to $\sqrt{2}$ only as $\alpha \to \infty$. Hence a superexponential budget is necessary and sufficient for vanishing nonadaptive excess as d grows.

We also quantify how a small number of adaptive rounds improves the fixed- dimensional exponent. A B-batch strategy chooses all queries in a batch before seeing any answer from that batch, while later batches can use earlier answers.

Theorem 3 (Batch upper hierarchy)**.** *Fix $d \geq 2$ and $B \geq 1$, and put $q_d = \binom{d+1}{2}$. There are constants $C_{d,B}, M_{d,B} < \infty$ such that, whenever $M = T - (B-1)q_d \geq M_{d,B}$,*

$$\mathrm{OPT}_d^{(B)}(T, \delta) - J_d\delta \leq C_{d,B}\delta M^{-2^B/(d-1)}.$$

The first batch is a spherical net. Each later batch queries all edges of one simplex, using the robust Jung refinement of Filmus et al. (2026). We do not claim that Theorem 3 is minimax optimal for intermediate B.

Proof idea. For fixed nonadaptive directions $V = \{v_1, \ldots, v_T\}$, define

$$\kappa(V) = \min_{u\in\mathbb{S}^{d-1}} \max_{i\leq T} |\langle u, v_i\rangle|.$$

This is the inradius of $\mathrm{conv}(\{\pm v_i\}_{i=1}^T)$. Every transcript has feasible diameter at most $2\delta/\kappa(V)$. Jung's theorem then gives error at most $J_d\delta/\kappa(V)$. Good spherical covers make $\kappa(V)$ large.

The converse needs more than a missed direction. A two-point ambiguity only gives a limiting error δ, not $J_d\delta$. Instead take a regular d-simplex of side 2δ. Rotate it so that none of its $\binom{d+1}{2}$ edge directions is too close to any query, then expand it. Every queried directional width remains at most 2δ. Answer each query with the midpoint of the extreme vertex projections. All $d + 1$ expanded vertices now produce one transcript, so one of them is at least the expanded circumradius from the estimate. A cap union bound finds the rotation. In fixed dimension it yields an avoided angle of order $T^{-1/(d-1)}$. With $T = \exp(\alpha d + o(d))$, the spherical-cap large-deviation exponent yields the exact threshold $\sqrt{1 - e^{-2\alpha}}$.

2 Model and geometric reduction

All strategies in this paper are deterministic, matching Filmus et al. (2026). A nonadaptive strategy chooses $v_1, \ldots, v_T \in \mathbb{S}^{d-1}$ before the game. The adversary knows the strategy, chooses $x^\star$, and returns any answers obeying Equation (1). The strategy then maps the transcript to an estimate. The minimax error is

$$\mathrm{OPT}_d^{\mathrm{na}}(T, \delta) = \inf_{V,A} \sup_{x^\star \in \mathbb{R}^d} \sup_{|r_i - \langle v_i, x^\star \rangle| \le \delta} \|A(r_1, \ldots, r_T) - x^\star\|_2.$$

The case $d = 1$ is degenerate. One query attains error δ, so we assume $d \geq 2$. Scaling gives $\mathrm{OPT}_d^{\mathrm{na}}(T, \delta) = \delta \, \mathrm{OPT}_d^{\mathrm{na}}(T, 1)$.

For a transcript r, let

$$\Phi(V, r) = \{x \in \mathbb{R}^d : |\langle v_i, x \rangle - r_i| \le \delta \text{ for all } i \le T\}. \tag{2}$$

For a bounded set K, let $\mathrm{rad}(K) = \inf_z \sup_{x \in K} \|x - z\|_2$. The optimal output for a fixed transcript is a Chebyshev center of its feasible set. Thus the error of a design is $\sup_r \mathrm{rad}(\Phi(V, r))$ over nonempty feasible sets.

Proposition 4 (Covering upper bound)**.** *For every V with $\kappa(V) > 0$,*

$$\sup_{r : \Phi(V,r) \neq \varnothing} \mathrm{rad}(\Phi(V, r)) \le \frac{J_d \delta}{\kappa(V)}.$$

Proof. Take $x, y \in \Phi(V, r)$ and put $u = (x - y)/\|x - y\|_2$. Some v_i satisfies $|\langle u, v_i \rangle| \geq \kappa(V)$. Feasibility gives $|\langle x - y, v_i \rangle| \le 2\delta$, hence $\|x - y\|_2 \le 2\delta/\kappa(V)$. Jung's theorem (Jung, 1901; Gruber, 2007) states

$$\mathrm{rad}(K) \le \sqrt{\frac{d}{2(d+1)}} \, \mathrm{diam}(K).$$

Substitution proves the claim. □

We next isolate the ambiguity construction used in both lower bounds. For a finite set K, write $\mathrm{width}_K(v) = \max_{x \in K} \langle v, x \rangle - \min_{x \in K} \langle v, x \rangle$.

Lemma 5 (Rotated-simplex transcript)**.** *Let $V \subset \mathbb{S}^{d-1}$ and let Δ be a regular d-simplex of side 2δ. Suppose a rotation R and $m \in (0, 1]$ satisfy*

$$|\langle v, Re \rangle| \le m \quad \textit{for every } v \in V \textit{ and every unit edge direction } e \textit{ of } \Delta.$$

Then the worst-case error of the design V is at least $J_d \delta / m$.

Proof. Let $K = m^{-1} R \Delta$. A directional width of a polytope is attained by a pair of vertices. Since every edge of Δ has length 2δ, the hypothesis gives $\mathrm{width}_K(v) \le 2\delta$ for every $v \in V$. Return

$$r_v = \tfrac{1}{2}\big(\max_{x \in K} \langle v, x \rangle + \min_{x \in K} \langle v, x \rangle\big).$$

Every vertex of K is consistent with every answer. The Chebyshev radius of these vertices is the circumradius $J_d \delta / m$. Therefore some vertex is at least that far from the design's output. Because V is fixed before the game, the adversary can choose this vertex as $x^\star$ at the outset and return the displayed common transcript. □

Lemma 5 captures why noncentral information fibers matter. The zero-answer fiber is centrally symmetric and yields only a two-point lower bound. The least favorable fiber here contains a regular simplex. This is also why general factor-two adaptivity bounds in optimal recovery do not settle the excess rate.

3 FIXED DIMENSION

For unit vectors, let $\rho(u,v) = \arccos|\langle u,v\rangle|$ be projective angular distance. A projective η-net has a point within distance η of every line. Standard volumetric covering estimates give the following facts for fixed $d \geq 2$:

$$\begin{aligned} &\text{there is a projective } \eta\text{-net of size at most } a_d\eta^{-(d-1)}, \\ &\mu_d\{u : \rho(u,e) \leq \eta\} \leq b_d\eta^{d-1} \quad \text{for } 0 < \eta \leq \eta_d. \end{aligned} \tag{3}$$

where μ_d is uniform spherical measure. Appendix A gives a short proof with explicit integral bounds.

Proof of Theorem 1. For the upper bound, choose T directions forming a projective net of radius $\eta \leq A_d T^{-1/(d-1)}$. Then $\kappa(V) \geq \cos\eta$. Proposition 4 and $1/\cos\eta = 1 + \Theta(\eta^2)$ give

$$\mathrm{OPT}_d^{\mathrm{na}}(T,\delta) \leq \frac{J_d\delta}{\cos\eta} \leq J_d\delta + C_d\delta T^{-2/(d-1)}.$$

For the lower bound, fix arbitrary queries V. Independently rotate a regular simplex Δ. Each of its $N_d = \binom{d+1}{2}$ unoriented edge directions is marginally uniform on projective space. A union bound and Equation (3) show

$$\Pr\left\{\rho(Re,v) < \eta \text{ for some edge } e, v \in V\right\} \leq N_d T b_d \eta^{d-1}.$$

Choose $\eta = c_d' T^{-1/(d-1)}$ so that the right side is below one. Some rotation then has every edge-query angle at least η. Lemma 5 with $m = \cos\eta$ gives

$$\mathrm{OPT}_d^{\mathrm{na}}(T,\delta) \geq \frac{J_d\delta}{\cos\eta} \geq J_d\delta + c_d\delta T^{-2/(d-1)}.$$

The Taylor inequalities used above hold after increasing T_d if needed. $\square$

The exponent has a simple geometric origin. Covering a $(d-1)$-dimensional projective sphere gives angular resolution $T^{-1/(d-1)}$. Both the secant loss in the upper bound and the safe expansion of the hidden simplex are quadratic in that angle.

4 THE EXPONENTIAL-BUDGET PHASE CURVE

The high-dimensional result uses a classical spherical-cap exponent. If U is uniform on $\mathbb{S}^{d-1}$ and $m \in (0,1)$ is fixed, define

$$p_d(m) = \Pr\{\langle U, e_1\rangle \geq m\}, \qquad I(m) = -\tfrac{1}{2}\log(1-m^2).$$

Since U_1^2 has a $\mathrm{Beta}(1/2,(d-1)/2)$ distribution,

$$\lim_{d\to\infty} \frac{1}{d}\log p_d(m) = \tfrac{1}{2}\log(1-m^2) = -I(m). \tag{4}$$

The upper bound also needs the matching covering exponent.

Lemma 6 (Spherical covering exponent)**.** *For fixed $m \in (0,1)$, $\mathbb{S}^{d-1}$ can be covered by $\exp(dI(m)+o(d))$ one-sided caps $\{u : \langle u,v\rangle \geq m\}$.*

Proof. The cap has normalized area $p_d(m)$. The spherical covering theorem of Böröczky & Wintsche (2003), sharpened by Dumer (2007), gives a cover of density at most $O(d\log d)$ for every acute cap. Thus the number of caps is at most $O(d\log d)/p_d(m)$. Equation (4) proves the claim. This exponent is also the classical asymptotic inradius law for few-vertex polytopes (Bárány & Füredi, 1988). $\square$

Proof of Theorem 2. Let

$$m_\alpha = \sqrt{1 - e^{-2\alpha}}, \qquad I(m_\alpha) = \alpha.$$

For any fixed $m < m_\alpha$, Lemma 6 supplies at most T query directions for all sufficiently large d. These one-sided caps cover the sphere, so $\kappa(V) \geq m$. Proposition 4 yields

$$\limsup_{d\to\infty} \frac{\mathrm{OPT}_d^{\mathrm{na}}(T,\delta)}{\delta} \leq \frac{\sqrt{2}}{m}.$$

Letting $m \uparrow m_\alpha$ gives the desired upper limit.

For the converse, fix arbitrary T queries and choose a uniform random rotation of a regular simplex. For any fixed $m > m_\alpha$, a union bound over $N_d = \binom{d+1}{2}$ edges and T queries gives

$$\Pr\left\{|\langle Re, v_i\rangle| > m \text{ for some } e, i\right\} \leq 2N_d T p_d(m) = \exp\{d(\alpha - I(m)) + o(d)\} \longrightarrow 0.$$

Hence a rotation with all correlations at most m exists. Lemma 5 gives

$$\liminf_{d\to\infty} \frac{\mathrm{OPT}_d^{\mathrm{na}}(T,\delta)}{\delta} \geq \frac{\sqrt{2}}{m}.$$

Letting $m \downarrow m_\alpha$ matches the upper limit. $\square$

The argument uses classical optimal spherical coverage on the upper side and a different object on the lower side. The lower witness is not the uncovered line itself. It is one rotation of all simplex edges that simultaneously avoids the queries. The polynomial number N_d of edges disappears in the exponential scale, which is why the two exponents match exactly.

5 A HIERARCHY FROM ADAPTIVE BATCHES

The fixed-dimensional proof also clarifies how interaction enters. We use one result from the adaptive analysis of Filmus et al. (2026). In the normalization $\delta = 1/2$, if a feasible region has radius at least $J_d/2$ and diameter at most $1 + \beta$, their robust Jung theorem localizes its radius- witnessing core to a $C_d\beta$ neighborhood of a regular unit simplex. Querying all q_d edge directions of that simplex reduces the neighborhood radius from γ to at most $A_d\gamma^2$. All q_d directions depend only on the previous feasible region, so they form one batch.

Proof of Theorem 3. Use M directions in the first batch to form a projective net of radius $O_d(M^{-1/(d-1)})$. Proposition 4, before applying Jung's radius factor, gives feasible diameter

$$2\delta\big(1 + O_d(M^{-2/(d-1)})\big).$$

After scaling to $\delta = 1/2$, robust Jung localization gives a simplex neighborhood of radius $\gamma_0 = O_d(M^{-2/(d-1)})$. For $b = 1, \ldots, B-1$, spend one batch of q_d queries on the current simplex edges. The refinement recurrence is $\gamma_b \leq A_d\gamma_{b-1}^2$. For fixed d, B and sufficiently large M, iteration yields

$$\gamma_{B-1} \leq C_{d,B} M^{-2^B/(d-1)}.$$

The feasible radius is at most the regular-simplex radius plus γ_{B-1} in normalized units. Rescaling proves the theorem. $\square$

For fixed B, Theorem 3 implies query complexity $O_{d,B}(\epsilon^{-(d-1)/2^B}) + (B-1)q_d$ for excess $\epsilon\delta$. The case $B = 1$ matches Theorem 1. Once the number of batches grows, the repeated squaring transitions to the adaptive $\log\log(1/\epsilon)$ law. A matching minimax characterization for every intermediate batch budget remains open.

6 RELATED WORK AND LIMITATIONS

Linear reconstruction and adaptive sensing. Our direct predecessor is Filmus et al. (2026), which introduced this continuous linear-query game and proved the adaptive results summarized above. It builds on approximate distance reconstruction (Moran & Nesterova, 2025). Adaptive sensing more commonly assumes a structured signal and stochastic noise (Haupt et al., 2009; Arias-Castro et al., 2013). Those goals and noise models differ from worst-case localization of an unrestricted point.

Optimal recovery and bounded-error estimation. Optimal recovery studies algorithms from exact or inaccurate information (Traub et al., 1988; Osipenko, 2019). Recent sharp results construct optimal linear recovery maps for prescribed operators under hyperellipsoidal model classes and ℓ_1 or ℓ_2 data uncertainty (Foucart & Liao, 2024). Set-membership and Chebyshev-center estimators likewise recover from intersections of deterministic constraint sets (Eldar et al., 2008). This literature motivates our feasible-radius formulation. It typically fixes the information operator or imposes a bounded model class. The present question optimizes unit directions, compares answer- dependent and oblivious designs, and asks for the excess above a nonzero Jung limit.

Spherical covering. The projective-net estimates behind our upper bounds are classical. Sharp covering densities and equivalent inradius estimates appear in Böröczky & Wintsche (2003); Dumer (2007); Bárány & Füredi (1988); Bárány & Simányi (2005). We use their covering exponent rather than claim a new polytope approximation result. The reconstruction lower bound is different: it must place an expanded simplex inside one information fiber, which requires simultaneous avoidance by every edge direction.

Limitations. We follow the predecessor's deterministic minimax convention. The results also cover public randomization when the realized design is visible before the secret is selected. Private randomized nonadaptive designs under alternative move orders require a separate formulation. Our estimator is information-theoretic because it outputs an exact Chebyshev center. Computing that center is generally nontrivial, although standard convex optimization gives approximations. Finally, Theorem 3 is an upper interpolation, not a matching batch lower bound.

7 Conclusion

Nonadaptive noisy linear reconstruction has the same infinite-query Jung limit as adaptive reconstruction, but it approaches that limit at a fundamentally different speed. A spherical net gives the optimal polynomial exponent, while a rotated regular simplex supplies the matching ambiguity. In high dimension, the same construction converts the classical cap exponent into an exact exponential-budget minimax curve. These results resolve the nonadaptive problem posed by Filmus et al. (2026) and isolate the geometric value of interaction: adaptive batches repeatedly square the residual simplex uncertainty.

Reproducibility statement

All results are proved analytically. The supplement includes a deterministic diagnostic that constructs regular simplices, searches rotations, verifies the common-transcript inequalities, and checks finite-dimensional convergence of the spherical-cap exponent. These computations are sanity checks and are not used as proof steps.

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A SPHERICAL CAPS AND PROJECTIVE NETS

This appendix records the cap estimates used in Sections 3 and 4. They are included to make the dependence on the two asymptotic regimes explicit.

A.1 FIXED DIMENSION

Let $C_d(\eta) = \{u \in \mathbb{S}^{d-1} : \arccos|\langle u, e_1\rangle| \le \eta\}$ be a projective cap. For $0 < \eta < \pi/2$, normalized surface measure gives

$$\mu_d(C_d(\eta)) = \frac{2\int_0^\eta \sin^{d-2}\theta\, d\theta}{\int_0^\pi \sin^{d-2}\theta\, d\theta}. \tag{5}$$

For fixed d, $\sin\theta \le \theta$ yields

$$\mu_d(C_d(\eta)) \le c_d \eta^{d-1}.$$

The reverse inequality with another constant holds for $0 < \eta \le \eta_d$, since $\sin\theta \ge 2\theta/\pi$ on $[0, \pi/2]$.

Take a maximal set of projective points separated by more than η. Maximality makes it an η-net. Its projective caps of radius $\eta/2$ are disjoint. The lower cap-volume estimate therefore bounds its cardinality by $a_d\eta^{-(d-1)}$. This proves both assertions in Equation (3).

A.2 High-dimensional cap exponent

For U uniform on $\mathbb{S}^{d-1}$, the first coordinate has density

$$f_d(t) = c_d(1-t^2)^{(d-3)/2}, \qquad c_d = \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d-1)/2)}. \tag{6}$$

Stirling's formula gives $\log c_d = o(d)$. For fixed $m \in (0,1)$,

$$p_d(m) = \int_m^1 f_d(t)\,dt \le c_d(1-m)(1-m^2)^{(d-3)/2}.$$

This proves

$$\limsup_{d\to\infty} \frac{1}{d}\log p_d(m) \le \tfrac{1}{2}\log(1-m^2).$$

For every fixed $\epsilon \in (0, 1-m)$,

$$p_d(m) \ge \epsilon c_d\big(1-(m+\epsilon)^2\big)^{(d-3)/2}.$$

First let $d \to \infty$, then let $\epsilon \downarrow 0$. This gives the matching lower limit and proves Equation (4).

For completeness, we spell out why the covering theorem supplies the same exponent. A cap $\{u : \langle u, v\rangle \ge m\}$ has acute angular radius $\phi = \arccos m$. It can be represented as the intersection of a sphere of radius $r = 1/\sin\phi > 1$ with a unit Euclidean ball. Theorem 1 of Dumer (2007), with sphere dimension $d-1$, gives a cover whose density is at most $O(d\log d)$, uniformly over $r > 1$. Dividing this density by the normalized cap area $p_d(m)$ bounds the number of caps by

$$\frac{O(d\log d)}{p_d(m)} = \exp(dI(m) + o(d)).$$

B Details for the nonadaptive lower bounds

We verify the simplex normalizations and the move order used in Lemma 5.

B.1 Regular-simplex normalization

Let $z_0, \dots, z_d \in \mathbb{R}^d$ obey

$$\langle z_i, z_i\rangle = \frac{d}{d+1}, \qquad \langle z_i, z_j\rangle = -\frac{1}{d+1} \quad (i \ne j).$$

These are centered regular-simplex vertices with side length $\sqrt{2}$ and circumradius $\sqrt{d/(d+1)}$. Multiplying each vertex by $\sqrt{2}\delta$ gives side 2δ and circumradius

$$\sqrt{2}\delta\sqrt{\frac{d}{d+1}} = J_d\delta.$$

After rotation and scaling by $1/m$, the circumradius is $J_d\delta/m$.

For any finite vertex set K and direction v,

$$\mathrm{width}_K(v) = \max_{x,y\in K}\langle v, x-y\rangle.$$

Every difference of two regular-simplex vertices is an edge vector. Thus the edge-correlation condition in Lemma 5 implies $\mathrm{width}_{m^{-1}R\Delta}(v) \le 2\delta$.

B.2 One transcript and one secret

Fix a deterministic nonadaptive algorithm. Its complete query set V is a known function of the algorithm, independent of the secret and answers. The rotation in Lemma 5, the scaled simplex K, the midpoint answers, and the algorithm's output on those answers can therefore all be determined before the interaction begins. Choose a farthest vertex

$$x^\star \in \arg\max_{x\in K} \|x - A(r)\|_2.$$

The midpoint construction ensures $|r_v - \langle v, x^\star \rangle| \le \delta$ for every query. Hence this is a legal choice of one secret at the start of the game, not a post hoc switch between secrets. Since the minimum enclosing ball of a regular simplex is its circumsphere,

$$\|x^\star - A(r)\|_2 \ge \mathrm{rad}(K) = \frac{J_d \delta}{m}.$$

B.3 Why random rotation is enough

Let E be the set of $N_d = \binom{d+1}{2}$ unoriented unit edge directions of a reference simplex. If R is Haar uniform on the orthogonal group, then for each fixed $e \in E$, Re is uniform on the sphere. The different rotated edges are dependent, but both lower bounds use only the union bound. In fixed dimension,

$$\Pr\{\rho(Re, v_i) < \eta \text{ for some } e, i\} \le N_d T \mu_d(C_d(\eta)).$$

In the exponential-dimensional regime,

$$\Pr\{|\langle Re, v_i \rangle| > m \text{ for some } e, i\} \le 2 N_d T p_d(m).$$

Whenever the displayed upper bound is below one, at least one deterministic rotation has the required simultaneous avoidance property.

C Batch refinement details

We give a more explicit derivation of Theorem 3 from the robust Jung machinery of Filmus et al. (2026). Their proof uses noise $\delta = 1/2$, so queried slabs have width one. Write

$$\eta_M = a_d M^{-1/(d-1)}, \qquad \beta_M = \frac{1}{\cos \eta_M} - 1 = O_d(M^{-2/(d-1)}).$$

After querying the first-batch net, every feasible region Φ_0 has $\mathrm{diam}(\Phi_0) \le 1 + \beta_M$.

The robust Jung theorem in the witness-core form states that, whenever β_M is below a dimension-dependent threshold and the feasible radius has not already dropped below the Jung benchmark, there is a regular unit simplex Δ_0 such that

$$\mathrm{Witness}(\Phi_0) \subseteq \Delta_0 + B(C_d \beta_M).$$

The witness core has the same Chebyshev radius as the feasible region. Put $\gamma_0 = C_d \beta_M$.

Suppose at the start of a later batch that

$$\mathrm{Witness}(\Phi_b) \subseteq \Delta_b + B(\gamma_b),$$

where Δ_b is a regular unit simplex. Query all its $q_d = \binom{d+1}{2}$ unit edge directions in parallel. The refinement argument in the proof of Main Theorem 2 of Filmus et al. (2026) gives

$$\mathrm{diam}(\mathrm{Witness}(\Phi_{b+1})) \le 1 + 4\gamma_b^2.$$

A second application of robust Jung gives another regular unit simplex and

$$\gamma_{b+1} \le 4 C_d \gamma_b^2.$$

For fixed d, B, take M large enough that every iterate remains below the robust-Jung threshold. Induction yields

$$\gamma_{B-1} \le C_{d,B} \beta_M^{2^{B-1}} \le C'_{d,B} M^{-2^B/(d-1)}.$$

The radius of $\Delta_{B-1} + B(\gamma_{B-1})$ is at most its simplex circumradius plus γ_{B-1}. This proves the normalized result. General δ follows by scaling.

If the radius ever drops below the Jung benchmark during the argument, the desired upper bound is already true after enlarging $C_{d,B}$. Thus the stated case distinction in the robust theorem causes no gap.

D Endpoint consequences of the phase curve

Theorem 2 is deliberately stated for a fixed $\alpha \in (0, \infty)$, where all cap estimates are uniform after fixing a small margin around m_α. Two endpoint consequences follow without a uniform large-deviation claim.

First, if $\log T/d \to \infty$, fix any $m < 1$. Lemma 6 shows that the sphere can eventually be covered using at most T caps of threshold m. Proposition 4 gives

$$\limsup_{d\to\infty} \frac{\mathrm{OPT}_d^{\mathrm{na}}(T,\delta)}{\delta} \le \frac{\sqrt{2}}{m}.$$

Let $m \uparrow 1$. The universal Jung lower limit from Filmus et al. (2026) gives equality with $\sqrt{2}$.

Second, if $\log T = o(d)$, nonadaptive strategies are a subset of adaptive strategies. The subexponential lower bound of Filmus et al. (2026) therefore implies

$$\mathrm{OPT}_d^{\mathrm{na}}(T,\delta) - J_d\delta \longrightarrow \infty.$$

The limiting curve in Theorem 2 is consistent with both endpoints.

E Computational sanity checks

The file `verify_geometry.py` performs two independent diagnostics.

First, for $d \in \{2, 3, 5\}$ it constructs centered regular-simplex coordinates, draws a fixed nonadaptive query set, searches over seeded random rotations, and chooses the rotation with the smallest maximum edge-query correlation m. It then expands the simplex by $1/m$, forms every midpoint response, and checks

$$\max_i \mathrm{width}_K(v_i) \le 2\delta, \qquad \max_{x\in K,i} |\langle v_i, x\rangle - r_i| \le \delta, \qquad \mathrm{rad}(K) = \frac{J_d\delta}{m}.$$

All inequalities pass to floating-point tolerance 10^{-10}.

Second, it evaluates the exact two-sided cap tail using $U_1^2 \sim \mathrm{Beta}(1/2, (d-1)/2)$. For several values of α and $d \le 500$, it solves for the finite-dimensional correlation whose cap probability is $e^{-\alpha d}$ and compares it with $\sqrt{1 - e^{-2\alpha}}$. The absolute discrepancy decreases monotonically in every tested sequence. The generated CSV files contain the raw values.

These checks are not invoked in any proof. They guard the edge-length, noise-width, and beta-tail normalizations that are easiest to misstate.